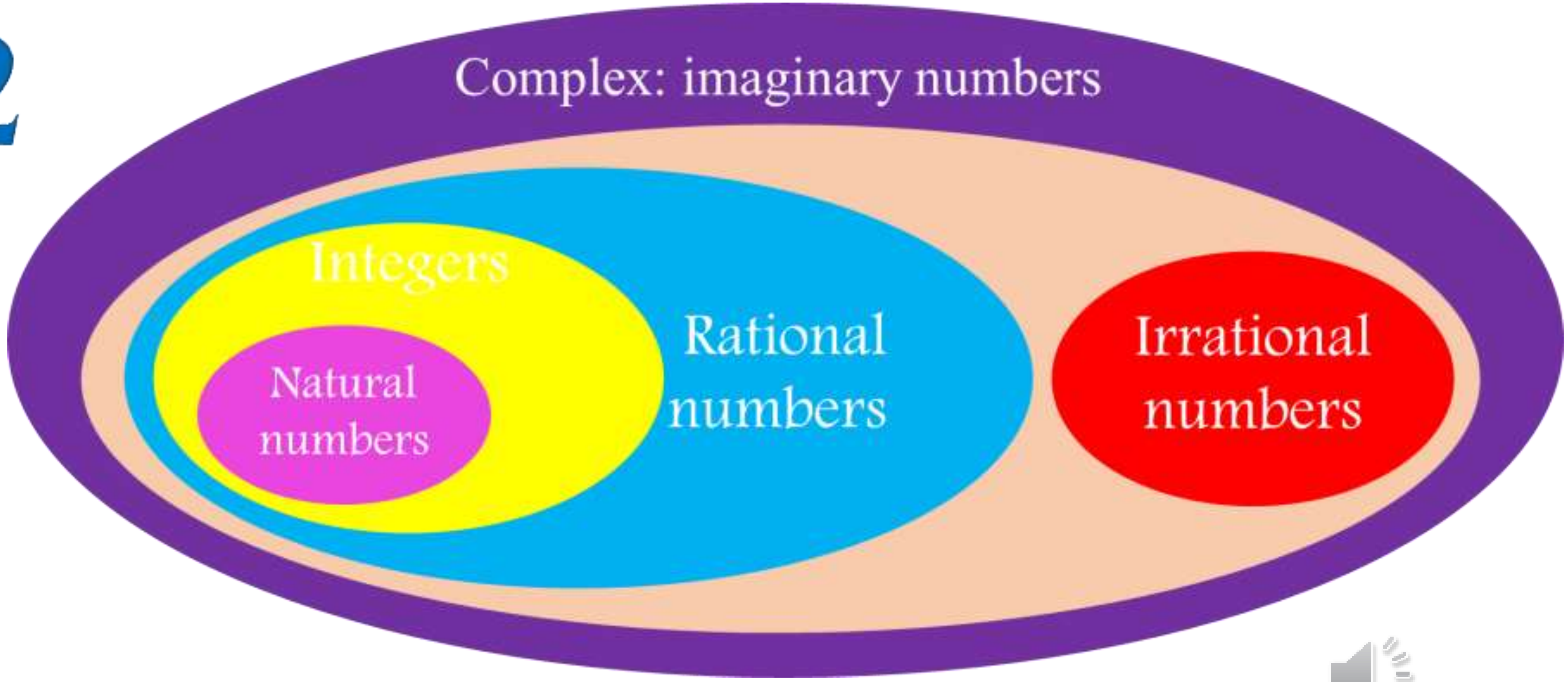
Complex Numbers

Part 2



Reciprocal of a non zero complex number

$z = a + bi$ is a complex number

The reciprocal of z is $\frac{1}{z} = \frac{\bar{z}}{a^2+b^2}$.

$$z \times \bar{z} = a^2 + b^2$$

$$z = \frac{a^2+b^2}{\bar{z}}$$

$$\frac{1}{z} = \frac{1}{\frac{a^2+b^2}{\bar{z}}} = \frac{\bar{z}}{a^2+b^2}$$

Reciprocal of a non zero complex number

$z = a + bi$ is a complex number

The reciprocal of z is $\frac{1}{z} = \frac{\bar{z}}{a^2+b^2}$.

Example:

$z = 1 + 2i$

$\frac{1}{z} = \frac{1-2i}{1^2+2^2} = \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i$

$z = \frac{2}{3}i$

$\frac{1}{z} = \frac{-\frac{2}{3}i}{\left(\frac{2}{3}\right)^2} = \frac{-\frac{2}{3}i}{\frac{4}{9}} = -\frac{3}{2}i$

Reciprocal of a non zero complex number

$$\overline{\left(\frac{z}{z'}\right)} = \frac{\bar{z}}{\overline{z'}}$$

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$$

$$z' = \frac{z'}{z} \times z$$

$$\overline{z'} = \overline{\frac{z'}{z} \times z} = \overline{\left(\frac{z'}{z}\right)} \times \bar{z}$$

$$\overline{\left(\frac{z'}{z}\right)} = \frac{\overline{z'}}{\bar{z}}$$

$$\overline{\left(\frac{1}{z}\right)} = \frac{\bar{1}}{\bar{z}} = \frac{1}{\bar{z}}$$

Example:

$z = 1 + 2i \quad ; \quad z' = 2 + i$

$$\overline{\left(\frac{z}{z'}\right)} = \frac{1-2i}{2-i}$$

Reciprocal of a non zero complex number

$Re(z) = \frac{1}{2}(z + \bar{z})$ and $Im(z) = \frac{1}{2i}(z - \bar{z})$

$z = a + bi$

$\bar{z} = a - bi$

$z + \bar{z} = 2a$

$a = \frac{z+\bar{z}}{2}$

$Re(z) = \frac{1}{2}(z + \bar{z})$

$z = a + bi$

$\bar{z} = a - bi$

$z - \bar{z} = 2bi$

$b = \frac{z-\bar{z}}{2i}$

$Re(z) = \frac{1}{2i}(z - \bar{z})$

Reciprocal of a non zero complex number

z is a real number if and only if $z = \bar{z}$
z is pure imaginary if and only if $z = -\bar{z}$

z is real so $Im(z) = 0$
$z = \bar{z} = a$

z is pure imaginary so $Re(z) = 0$
$z = bi$ so $\bar{z} = -bi$
$z = -\bar{z}$

Reciprocal of a non zero complex number

Given $z = a + bi$ and $z' = a' + b'i$

To write $\frac{z}{z'}$ in algebraic form $(x + iy)$ multiply the fraction by $\frac{\bar{z}'}{\bar{z}'}$

$$\frac{z}{z'} = \frac{z}{z'} \times \frac{\bar{z}'}{\bar{z}'} = \frac{z\bar{z}'}{z'\bar{z}'} = \frac{aa' - b(-b') + i(a(-b') + ba')}{a'^2 + b'^2} = \frac{aa' + bb' + i(-ab' + ba')}{a'^2 + b'^2}$$

Which is in the form of $x + iy$ where:

$x = \frac{aa' + bb'}{a'^2 + b'^2}$ and $y = \frac{-ab' + ba'}{a'^2 + b'^2}$

Reciprocal of a non zero complex number

Given $z = a + bi$ and $z' = a' + b'i$

To write $\frac{z}{z'}$ in algebraic form $(x + iy)$ multiply the fraction by $\frac{\bar{z'}}{\bar{z'}}$

Example:

$$\frac{1+2i}{1-i} = \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i+2i+2i^2}{1^2+1^2} = \frac{-1+3i}{2}$$

Square root of a complex number

We call a square root of a complex number z every complex number u such that $u^2 = z$.

Example:

$1 + i$ is a square root of $2i$ since $(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

Square root of a complex number

Every complex number has two opposite square root in $\mathbb{C}$.

Example:

$z = 5 - 12i$

$u = a + bi$ is a square root of z, so $u^2 = z$

$(a + bi)^2 = 5 - 12i$

$a^2 + 2abi - b^2 = 5 - 12i$

By comparing, $a^2 - b^2 = 5$ and $2ab = -12$

$2ab = -12$; $a = -\frac{12}{2b} = -\frac{6}{b}$

$a^2 - b^2 = 5$ so, $\frac{36}{b^2} - b^2 = 5$; $\frac{36-b^4}{b^2} = 5$ therefore $36 - b^4 = 5b^2$

$b^4 + 5b^2 - 36 = 0$

Let $t = b^2$; $t^2 = b^4$ so, $t^2 + 5t - 36 = 0$

Square root of a complex number

Every complex number has two opposite square root in $\mathbb{C}$.

Example:

$t^2 + 5t - 36 = 0$

$\Delta = b^2 - 4ac = 25 - 4(1)(-36) = 25 + 144 = 169$

$t_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-5-13}{2} = -9 \quad ; \quad t_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-5+13}{2} = 4$

For $t = -9$; $b^2 = -9$ impossible

For $t = 4$; $b^2 = 4$; $b = 2$ $or\ b = -2$

But $a = -\frac{6}{b}$

So $a = -\frac{6}{2} = -3$ or $a = -\frac{6}{-2} = 3$

Therefore the square roots of $z = 5 - 12i$ are $-3 + 2i$; $3 - 2i$

Quadratic equation

Any equation in the form of $A^2 + B^2 = 0$ has:

- No real roots in IR
- Two distinct roots in $\mathbb{C}$: $A^2 = -B^2 = B^2 i^2$ so $A = Bi \quad or \quad A = -Bi$

Example:

$x^2 = -1$

$x^2 = -1 = i^2$ so the roots are $x = i \quad or \quad x = -i$

$(x-1)^2 = -4$

$(x-1)^2 = 4i^2$

$x - 1 = 2i \quad or \quad x - 1 = -2i$

$x = 1 + 2i \qquad\qquad x = 1 - 2i$

Quadratic equation

Consider the quadratic equation $ax^2 + bx + c = 0 \;\; a \neq 0$ such that $\Delta < 0$
The equation in this case has two roots in $\mathbb{C}$ and they are conjugate complex numbers.

$\Delta < 0$

so $\Delta = ki^2$ where k is a positive number

$$z_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-b-\sqrt{k}i}{2a}$$

$$z_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-b+\sqrt{k}i}{2a}$$

$$Re(z_1) = Re(z_2)$$

$$Im(z_1) = -Im(z_2)$$

So z_1 & z_2 are conjugates.

Quadratic equation

Example:

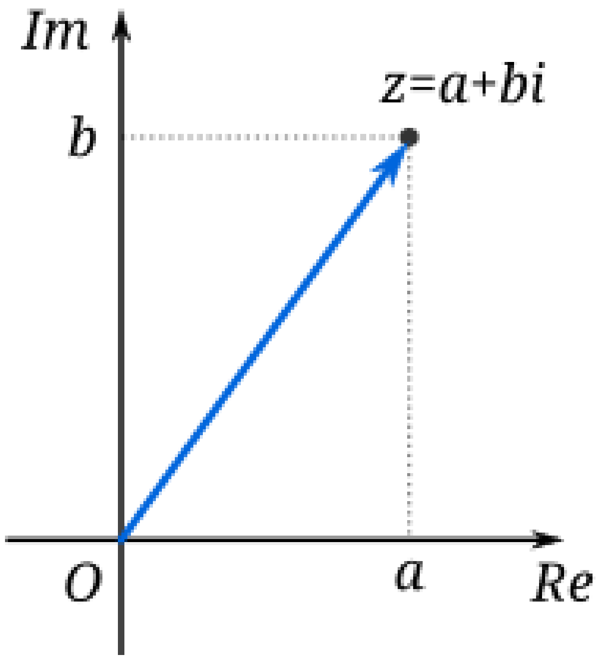
$z^2 + 3z + 4 = 0$

$\Delta = b^2 - 4ac = 9 - 16 = -7 = 7i^2$

$z_1 = \frac{-3-\sqrt{7}i}{2}$; $z_2 = \frac{-3+\sqrt{7}i}{2}$

Graphical representation

A complex number z can be identified with the ordered pair (Re(z);Im(z)) which can be the coordinates of a point in the system of coordinates.
(x'x) is used to display the real part.
(y'y) is used to display the imaginary part.



z is called the affix of the point.

A(z_A) and B(z_B) are two points. $z_{\overrightarrow{AB}} = z_B = z_A$
Example: $A(1+i)$ and B$(2+3i)$
$z_{\overrightarrow{AB}} = z_B - z_A = 2 + 3i - 1 - i = 1 + 2i$

Application

Answer with true or false and justify.

❶ the reciprocal of $\frac{2}{3}i$ is $\frac{3}{2}i$

False, $\frac{1}{\frac{2}{3}i} = \frac{3}{2i} \times \frac{-2i}{-2i} = -\frac{6i}{-4i^2} = -\frac{6i}{4} = -\frac{3}{2}i$

Application

Answer with true or false and justify.

❷ the algebraic form of $z = \frac{2}{3+i}$ is $\frac{3-i}{2}$

False,

$$z = \frac{2}{3+i} = \frac{2}{3+i} \times \frac{3-i}{3-i} = \frac{6-2i}{9-i^2} = \frac{6-2i}{10} = \frac{3-i}{5}$$

Application

Answer with true or false and justify.

❸ $z^2 + 3z + 1 = 0$ has real roots

True,

$\Delta = b^2 - 4ac = 9 - 4(1)(1) = 5$

$z_1 = \frac{-3-\sqrt{5}}{2}$ and $z_2 = \frac{-3+\sqrt{5}}{2}$ which are real numbers

Application

Answer with true or false and justify.

❹ the points with affixes $i; -i\ ; 3i$ and $\frac{i}{2}$ are collinear.

True, since:

The 4 affixes are pure imaginary, so the 4 points belong to (yAxis).

Application

Answer with true or false and justify.

❺ if A$(-1-3i)$ and B$(2-5i)$ then $\overrightarrow{AB}(-3+2i)$

False, since:

$z_{\overrightarrow{AB}} = z_B - z_A = 2-5i-(-1-3i) = 2-5i+1+3i = 3-2i$

Application

Answer with true or false and justify.

❻ the two points of affixes z and $\bar{z}$ are symmetric with respect to (xAxis)

True, since:

Suppose that M(z) and N($\bar{z}$)

$Re(z) = Re(\bar{z})$ so $x_M = x_N$

And $Im(z) = -Im(\bar{z})$ so $y_M = -y_N$